

Fuzzy Transitions from Quantum to Classical Mechanics and New Phenomena of Mesoscopic Objects* **

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A new “phase invariant” equation of motion for both microscopic and macroscopic objects is proposed. It reduces to the probabilistic wave equation for small masses and the deterministic classical equation for large masses. The motions of mesoscopic objects and fuzzy transitions between quantum and classical mechanics are discussed on the basis of the generalized equation. Experimental tests of new predictions are discussed.

1. Introduction

We know that microscopic objects are described by the Schroedinger equation

$$i\hbar\partial\Psi/\partial t = [(-i\hbar\nabla)^2/2m + V]\Psi \quad (1)$$

and are characterized by a probabilistic behavior, while the macroscopic objects obey the deterministic Newtonian equation of motion

$$\mathbf{F} = m\mathbf{a} \quad (2)$$

or, equivalently, the Hamilton-Jacobi equation

$$-\partial S/\partial t = (\nabla S)^2/2m + V. \quad (3)$$

We stress that the mass rather than the size of an object appears in the basic equation. It appears reasonable to assume that the deterministic behavior $\mathbf{r}(t)$ is necessary to guarantee the macroscopic definiteness of the apparatus and to formulate the quantum-mechanical concepts precisely. In this sense, both the probabilistic behavior $\Psi(\mathbf{r}, t)$ and the deterministic behavior $\mathbf{r}(t)$ should be present in a complete theory for the physical world in general. This view has been discussed in the literature [1, 2].

To explore this viewpoint further, one may ask: What is the behavior of the physical objects which are neither microscopic nor macroscopic? In other words, what is the basic equation of motion for “mesoscopic

objects”? Is there a gradual transition from quantum to classical mechanics? These problems are discussed on the basis of local gauge (or phase) symmetry. For a particle with an arbitrary mass m , we write the wave function in the form

$$\Psi \exp[i f S(\mathbf{r}, t)], \quad f = \text{constant},$$

where S is to be related to the classical energy E_C and momentum \mathbf{p}_C , e.g., $\mathbf{p}_C = \nabla S$, so that the particle has the probabilistic behavior for small m and the deterministic behavior for large m . In this sense, we have both $\Psi(\mathbf{r}, t)$ and $\mathbf{r}(t)$ in the theory. To accomplish this, the quantity f should depend on the mass m .

For definiteness, we shall assume a new mass scale M which characterizes the masses of mesoscopic objects, so that $m/M \ll 1$ and $m/M \gg 1$ denote respectively microscopic and macroscopic objects [3]. The value of M has to be determined by experiments in the future. But we can estimate its upper limit based on phenomena related to the Brownian motion of very small uniform latex particles (which are much smaller than pollens):

$$10^{-25} \text{ kg} \lesssim M \lesssim 10^{-19} \text{ kg}, \quad (4)$$

where the lower limit is estimated from molecular motions. We may remark that the mass scale M does not exist in the conventional quantum mechanics. However, if Klauder’s continuous representation in Hilbert space is used in quantum mechanics, it naturally allows a length scale which can be related to the new mass scale M [4, 5].

Now let us consider the microscopic relation

$$p x - x p = -i\hbar, \quad p = p_x. \quad (5)$$

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** With an Appendix by B. Zhou and J. P. Hsu.

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From a physical viewpoint, the existence of the macroscopic limit,

$$p x - x p = 0, \quad (6)$$

of (5) should be related to the physical limit $m/M \rightarrow 0$ rather than the “mathematical limit” $\hbar \rightarrow 0$, because the mass m of a particle can be varied experimentally while \hbar is a fixed constant of nature and cannot be changed in experiments. This observation enables us to discuss *gradual transition* from quantum to classical mechanics and fuzzy behavior of mesoscopic objects. These discussions are not possible within the framework of conventional quantum mechanics.

2. The Generalized Equation of Motion for Microscopic and Macroscopic Objects

The relations (5) and (6) and the fact that the quantum momentum \mathbf{p}_Q differs from the classical momentum $\mathbf{p}_C = m \mathbf{v}$ or ∇S suggest that *the true momentum for all masses* should be a linear combination of them [3]:

$$\mathbf{p}_T = Q \mathbf{p}_Q + C \mathbf{p}_C = -i \hbar Q \nabla + C \nabla S, \quad (7)$$

such that it reduces to the proper limits,

$$\mathbf{p}_T = \begin{cases} \mathbf{p}_Q, & m \rightarrow 0, \\ \mathbf{p}_C, & m \rightarrow \infty. \end{cases} \quad (8)$$

Thus, Q and C may be termed respectively the quantum fraction and the classical fraction, and they must be functions of the dimensionless parameter m/M . The limiting results in (8) naturally suggest that physical meaning of Q and C : Namely, when we observe the behavior of an object with an arbitrary mass m , we postulate to interpret that

- (a) Q is the probability of finding that it behaves like a quantum particle,
- (b) $C = 1 - Q$ is the probability of finding that it behaves like a classical particle.

For definiteness of discussions, let us assume that the quantum fraction Q and the classical fraction C are given by

$$Q = \exp(-m/M), \quad C = 1 - Q, \quad (9)$$

which has the desirable limiting properties for small and large masses. This is essentially just a simple example. Of course, there are other possibilities for the mass dependences of Q and C . If the quantum fraction

Q has the exponential form (9), the mass range for the mesoscopic objects is very narrow: $M/20 \lesssim m \lesssim 20 M$. The functional dependence can only be determined by experiments in the future.

Similar to the true momentum in (7), the true “energy” for an arbitrary mass m should take the form

$$E_T = Q E_Q + C E_C = Q (i \hbar \partial/\partial t) + C (-\partial S/\partial t). \quad (10)$$

Based on (7) and (10), we postulate a generalized equation of motion for a physical object having *an arbitrary mass m and moving in a potential V* :

$$E_T \Phi = [\mathbf{p}_T^2/(2mQ) + VQ] \Phi, \quad (11)$$

where

$$\begin{aligned} E_T &= i \hbar Q \partial/\partial t + C E_C, & \mathbf{p}_T &= -i \hbar Q \nabla + C \mathbf{p}_C, \\ E_C &= -\partial S/\partial t = \mathbf{p}_C^2/2m + V, & \mathbf{p}_C &= \nabla S. \end{aligned}$$

S is Hamilton’s principal function. The reason that the new equation (11) involves mQ and VQ rather than just the usual m and V is due to the fact that the terms involving E_Q and \mathbf{p}_Q are associated with the quantum fraction Q , as one can see in (7) and (10). This is necessary for the new equation (11) to be consistent with the Schroedinger equation (see (13) below). We may remark that the relativistic generalization of (11) is straightforward [3].

When the potential V does not involve time explicitly, we can write the solution Φ in the form

$$\Phi = \Psi \Phi_C = \Psi \exp[-i C S/Q \hbar], \quad (12)$$

$$S = -E_C t + \int_{T(\mathbf{r})}^{\mathbf{r}} \mathbf{p}_C(\mathbf{r}') \cdot d\mathbf{r}',$$

where $T(\mathbf{r})$ denotes that the integration is carried out over the actual trajectory of motion of the classical object and the end point of $T(\mathbf{r})$ is \mathbf{r} itself. The trajectory $T(\mathbf{r})$ is determined by the Hamilton-Jacobi equation in (11). One can verify that for any value of Q , the function Ψ satisfies the Schroedinger equation

$$i \hbar \partial \Psi / \partial t = [(-i \hbar \nabla)^2/2m + V] \Psi. \quad (13)$$

The normalization condition for the quantum wave function Ψ should be

$$\int \Psi^*(\mathbf{r}) \Psi(\mathbf{r}) d^3 \mathbf{r} = Q, \quad (14)$$

so that it is consistent with the idea of quantum fraction in (7). Thus, $\Psi^*(\mathbf{r}) \Psi(\mathbf{r}) d^3 \mathbf{r}$ is *the probability of finding the particle (nearby \mathbf{r}) which behaves like a quantum particle (i.e., described by the Schroedinger equation)*. As a result, the probabilistic nature of a

particle will gradually fade away as its mass increases (i.e., $C \rightarrow 1$). Thus, in the quantum limit we have the usual wave function

$$\Phi = \Psi, \quad Q \rightarrow 1, \quad (15)$$

On the other hand, in the classical limit, $C \rightarrow 1$, the classical phase

$$\Phi_C = \exp[-iCS/Q\hbar] \quad (16)$$

varies very rapidly, so that, if one calculates the amplitude $K(a, b)$ of a particle to go from (\mathbf{r}_a, t_a) to (\mathbf{r}_b, t_b) , only when a path and a nearby path all give the same phase (i.e., S does not vary) in the first approximation, do we have a nonvanishing amplitude. This path is the one given by the Hamilton-Jacobi equation in (11). Thus, we have the deterministic behavior of a classical object. We stress that this deterministic behavior is the large mass limit of the generalized equation (11) rather than that of the Schroedinger equation.

3. Local Phase Invariance and the Conservation Law

It is interesting to note that the generalized basic equation of motion (11) for all microscopic, mesoscopic and macroscopic objects turns out to be invariant under a local phase transformation: Namely, the physics implied by (11) does not change when the general wave function Φ and the Hamilton's principal function S transform as follows:

$$\begin{aligned} \Phi &\rightarrow \Phi' = \Phi \exp[iZ(\mathbf{r}, t)], \\ S &\rightarrow S' = S - Z(\mathbf{r}, t)\hbar/QC. \end{aligned} \quad (17)$$

This phase invariance can also be seen in the general form of the solution (12) for Φ in which Hamilton's principal function S appears only in the phase:

$$\Phi = \Psi e^{-iCS/Q\hbar} \rightarrow \Phi' = (\Psi e^{-iCS/Q\hbar}) e^{iZ} = (\Psi e^{-iCS'/Q\hbar}).$$

It can be shown that the probability density $\Phi^* \Phi$ satisfies the following equation:

$$\begin{aligned} \partial \Phi^* \Phi / \partial t + \nabla \cdot \mathbf{J} &= 0, \\ \mathbf{J} &= (\hbar/2im)(\Phi^* \nabla \Phi - \Phi \nabla \Phi^*) - (C/mQ) \mathbf{p}_C \Phi^* \Phi. \end{aligned} \quad (18)$$

Note that the second term in \mathbf{J} is related to the classical momentum, and that $\Phi^* \Phi$ is normalized to Q rather than unity. We may remark that the current density \mathbf{J} is consistent with (7) (and (27) below) and that it is not directly observable, so that the last extra term in \mathbf{J} will not contradict previous experiments. Also, the conservation law (18) is related to the phase invariant of the generalized equation (11).

4. Fuzzy Transitions from Quantum to Classical Mechanics

The generalized equation of motion (11) implies new physical results for the motion of mesoscopic objects. Let us consider the double slit experiment. The solution for a free mesoscopic objects is

$$\Phi = A \exp \left\{ -i(\omega - CE_C/\hbar Q)t + i \left[\mathbf{k} \cdot \mathbf{r} - (C\mathbf{p}_C/\hbar Q) \cdot \int_{\mathbf{r}(t)}^{\mathbf{r}} d\mathbf{r}' \right] \right\}. \quad (19)$$

In this experiment, a coherent beam is separated into two parts by a double slit but brought together again in an area called the "interference region". In this region, the wave function Φ can be written as

$$\begin{aligned} \Phi &= \Phi_0 \exp \left[-i(C/\hbar Q) \int_{\text{path 1}}^{\mathbf{r}} \mathbf{p}_C \cdot d\mathbf{r} \right] \\ &+ \Phi_0 \exp \left[-i(C/\hbar Q) \int_{\text{path 2}}^{\mathbf{r}} \mathbf{p}_C \cdot d\mathbf{r} \right]. \end{aligned} \quad (20)$$

We have the closed line integral along path 1 and then along path 2 in the opposite direction. Thus we have

$$\oint \mathbf{p}_C \cdot d\mathbf{r} = \oint (\nabla S) \cdot d\mathbf{r} = \int (\nabla \times \nabla S) \cdot \mathbf{n} da = 0, \quad (21)$$

where the last surface integral is over the area bounded by path 1 and path 2. Thus, there is no interference effect due to the phase related to Hamilton's principal function S . Now suppose the intensity of the particle beam and the particle momentum $\mathbf{p}_Q = \hbar \mathbf{k} = \mathbf{p}$ is fixed and one varies only the mass m of the particle in this experiment. We have the following results:

- (A) When $Q \cong 1$, one has the usual interference pattern on the screen with the intensity I .
- (B) When $Q \cong 0$, one has only two bright spots at a and b which correspond to the end points of classical trajectories passing through the two slits.
- (C) When $Q \cong 1/2$, one has the interference pattern on the screen with roughly half of the intensity, $I/2$, of that in the case (A). Also, there are two bright spots at a and b with half of the brightness of that in the case (B).

Thus we have seen that there is a fuzzy transition between quantum and classical mechanics as the particle varies from very a small mass to a very large mass.

If we consider a mesoscopic simple harmonic oscillator. The total energy for such a system turns out

to be

$$E_{Tn} = Q(n + 1/2)\hbar\omega + Ckx_{\max}^2/2. \quad (22)$$

When Q is not equal to unity, we have a non-integral energy gap. This can be tested by considering a mesoscopic object moving in a strong magnetic field: In the presence of the electromagnetic four-potential (A^0, \mathbf{A}) , the new Eq. (11) is changed by the following replacements:

$$\begin{aligned} i\hbar\partial/\partial t &\rightarrow i\hbar\partial/\partial t - eA^0/c, \quad V \rightarrow 0, \\ -i\hbar\nabla &\rightarrow -i\hbar\nabla - e\mathbf{A}/c, \\ -\partial S/\partial t &= (\nabla S)^2/2m \rightarrow (-\partial S/\partial t - eA^0/c) \\ &= (\nabla S - e\mathbf{A}/c)^2/2m. \end{aligned} \quad (23)$$

Suppose the charged mesoscopic particle with a mass m is moving on the x - y plane with $\mathbf{v} = (v_x, v_y, 0)$ and in a constant magnetic field in the z direction. We take the potentials to be

$$A^0 = 0, \quad \mathbf{A} = (0, eBx/c, 0). \quad (24)$$

As usual, we can write the Hamiltonian in the Schroedinger equation in the form of a simple harmonic oscillator and obtain the energy spectrum

$$QE_n = (n + 1/2)Q\hbar eB/mc. \quad (25)$$

Thus the true energy E_T of the mesoscopic charged particle is

$$\begin{aligned} E_T &= QE_n + CE_C = Q(n + 1/2)\hbar eB/mc \\ &+ Cm\mathbf{v}^2/2, \end{aligned} \quad (26)$$

where \mathbf{v} is its classical velocity. It is hoped that this will be tested experimentally in the future.

Finally, let us consider the scattering of a mesoscopic object by a one-dimensional square potential barrier with a height V_0 and thickness a . Suppose A , B and T are amplitudes of incident, reflected and transmitted wave functions Ψ 's, respectively. For the case $0 < E < V_0$ we have the following results:

$$\begin{aligned} \text{transmitted coefficient} &= |T/A|^2, \\ \text{reflected coefficient} &= |B/A|^2 + C, \end{aligned} \quad (27)$$

where $|B/A|^2 + |T/A|^2 = Q$ according to the normalization condition (14)*.

* *Note added in proof:* In the Sudarshan Workshop, Prof. Iwo Bialynicki-Birula informed the author that there is a precise measurements of Fresnel diffraction with slow neutrons carried out by R. Gähler, A. G. Klein and A. Zeilinger, (Phys. Rev. A **23**, 1611 (1981)) and that their result may be used to estimate the lower limit of the new mass scale M in the generalized equation of motion (11) for both microscopic and macroscopic objects. From equations (11), (12) and (13), one can see that the Fresnel diffraction (by an absorbing straight-edge) in this theory is the same as that in conventional quantum mechanics. There is no change in the distance between the abscissa of point P and that of the first maximum of the diffraction pattern, where the point P corresponds to the edge of the geometrical shadow of the diffracting straight edge, i.e., its projection upon the plane of observation. However, according to this theory, there is a discontinuity of intensity at point P ,

$$\Delta I/I_0 = C \approx m/M, \quad (28)$$

where we have used (9). The measurements of Gähler et al. do not lead to an estimate of M better than that in (4). However, if this type of experiment is carried out by using a heavy molecular beam, the new effect (28) will be larger and may be detected. Also, the interferometer experiment which measures fringe shifts would be a very sensitive test of this theory, provided that one can make a neutron beam and say, a molecular beam to produce the interference pattern. Unfortunately, this is technically difficult at present.

Acknowledgement

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- [4] J. R. Klauder, J. Math. Phys. **4**, 1055 (1963); **4**, 1058 (1963); **5**, 177 (1964); in *Path Integrals and Their Applications in Quantum, Statistical and Solid State Physics*, edited by G. J. Papadopoulos and J. T. Devreese (Plenum, New York, 1978) pp. 5-38.
- [5] In a generalized quantum mechanics based on a new assumption $(\Delta x)_{\min} = R > 0$ (i.e., a quantum particle's position by itself cannot be measured with unlimited accuracy, even if we do not measure its momentum at the

same time), Klauder's continuous representation in Hilbert space is very useful for "realizable" physical states because it excludes the "unrealizable" position eigenstates $\langle x |$. This very small length scale R could be related to M by the relation $M = \hbar/c R$. In this formalism, the coordinate x of a quantum particle becomes a fuzzy dy-

namical variable and we have a new picture of "fuzzy point particle". See J. P. Hsu and S. Y. Pei, Phys. Rev. A **37**, 1406 (1988); J. P. Hsu and Chagarn Whan, Phys. Rev. A **38**, 2248 (1988); J. P. Hsu, Nuovo Cimento B **89**, 14 (1985); **80**, 183 (1984).

Appendix

A proposed experiment for testing fuzzy transitions: The Penning Trap.

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There are various kinds of electromagnetic traps which can store charged particles for a long time for doing experiment. Such devices are ideally suited for investigating rare particles or small objects in the mesoscopic world. Protons and ions are ubiquitous; however, mesoscopic objects are as rare as viruses. It is not easy to capture mesoscopic objects with masses in the range $10^{-15} - 10^{-20}$ kg. In the physical world, it appears that very few natural objects have masses in this range.

Suppose we can catch a virus and command it to carry 0.01 million units of elementary charge. Let us consider an experimental test of the fuzzy transition properties of a charged virus moving in a constant magnetic field. As shown in (26), the total mesoscopic energy E_T consists of a continuous classical energy $m v^2/2$ and a discrete quantum energy $(n + 1/2) \hbar q B / (m c)$:

$$E_T = Q(n + 1/2) \hbar q B / (m c) + C m v^2/2, \quad (A 1)$$

where q and m are, respectively, charge and mass of the mesoscopic object. A feasible experiment is to use the Penning trap to store a charged mesoscopic object for a long time and observe its energy.

A charged mesoscopic object can be trapped in a device which consists of a uniform magnetic field $\mathbf{B} = B_0 \mathbf{z}$ along the z -axis and a quadrupole electrostatic potential of the form [1]

$$V(r, z) = -A_0(r^2 - 2z^2), \quad A_0 > 0, \quad (A 2)$$

in cylindrical frame. The equipment is immersed in a liquid-helium bath. The uniform magnetic field \mathbf{B} can be generated by a superconducting solenoid, and the potential $V(r, z)$ can be generated by two end caps and a ring electrode.

In this case, the classical kinetic energy of the mesoscopic object moving with a velocity v is $E_C = m v^2/2$. It can be shown that there are three independent classical motions: an axial oscillation along the magnetic field line, a circular drift across this field line around the z -axis, and the familiar cyclotron motion around the z -axis. The energy E_C can be shown to be determined by the amplitudes, the frequencies, and the phases of these motions [2].

The discrete quantum energy is given by [3]

$$E_Q = \hbar \omega'_c(n + 1/2) + \hbar \omega_z(k + 1/2) - \hbar \omega_m(s + 1/2); \quad n, k, s = 0, 1, 2, 3, \dots, \quad (A 3)$$

where the axial frequency ω_z , the magnetron frequency ω_m and the modified cyclotron frequency ω'_c are respectively given by

$$\begin{aligned} \omega_z &= (4|q|A_0/m)^{1/2}, \\ \omega_m &= \omega_c/2 - (\omega_c^2/4 - \omega_m^2/2)^{1/2}, \\ \omega'_c &= \omega_c/2 + (\omega_c^2/4 - \omega_m^2/2)^{1/2}, \\ \omega_c &= |q|B_0/(m c). \end{aligned} \quad (A 4)$$

The transition of energy of this mesoscopic object has two parts:

$$\delta E_C = \delta(m v^2/2) = \text{continuous}; \quad (A 5)$$

$$\delta E_Q = \hbar \omega'_c(n - n') + \hbar \omega_z(k - k') - \hbar \omega_m(s - s'); \quad (A 6)$$

which correspond to the discrete Landau level transitions in (A 1).

Let us consider the order of magnitude of the cyclotron frequency involved in the experiment. For a proton with the mass m_p moving in a magnetic field $B_0 = 1$ Tesla $= 10^4$ Gauss, the cyclotron frequency is

$$\omega_c = 1.5 \times 10^7 \text{ Hz}. \quad (A 7)$$

If a mesoscopic object has a mass $m = 10^8 m_p$ and carries a charge $q = 10^4 e$ moving in the same magnetic field B , we have a cyclotron frequency

$$\omega_{cm} = 9.6 \times 10^3 \text{ Hz}. \quad (A 8)$$

Such a frequency of long waves can be measured. We may remark that the stability condition [2]

$$\omega_c > 1.414 \omega_z \quad (A 9)$$

can be satisfied because the physical quantities q , B_0 and A_0 in the relations (A 4) are adjustable. For example, suppose A_0 is very small, 10^{-4} (statvolt/cm²), we obtain $\omega_z = 3.4 \times 10^3$ Hz from the first equation in (A 4) for the same values of q and B_0 .

We note that for mesoscopic objects with larger masses, $\sim 10^{-16}$ kg, it becomes more difficult to satisfy the stability condition (A 9). In this case, the electrodynamic trap invented by Wuerker, Shelton, and Langmuir is more convenient [4]. It depends upon

finding electrode configurations which give sinusoidally time varying forces whose strengths are proportional to the distance from a central origin. Such a trap can be used to suspend any charged particle, charged dusts or electrons, in dynamics equilibrium.

If the fuzzy transition property is detected, one can vary the mass of the mesoscopic object and measure the intensity of a specific frequency to test the mass-dependence of the quantum and the classical coefficients Q and C .

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